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SMALL-TIME EXPANSION OF WAVE MOTION GENERATED BY A SUBMERGED SPHERE

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A nonlinear problem of motion of a solid sphere near a free surface of an infinitely deep fluid is considered. For the case of motion with a constant acceleration starting from rest, the solution is studied using a small-time expansion. Expansion coefficients up to the fourth power inclusive are found for the free surface elevation and for the force acting on the sphere. The solutions for linear and nonlinear conditions on the free surface are compared.

Key words*: surface waves, submerged sphere, integral boundary equation.*

Introduction. The problem of motion of bodies under the surface of a heavy fluid has been studied by many authors. An approximation for a body moving under the free surface by a velocity-potential singularity was first proposed by Lamb [1] and was further used by Havelock in the problem of motion of a sphere in a deep fluid [2]. Wu and Taylor [3] considered the motion of a sphere using multipole expansion over the sphere surface. All the above-mentioned solutions were obtained within the framework of the linear theory for the case of steady motion. Tyvand and Miloh [4] considered the impulsively starting motion of a horizontally submerged cylinder commencing from rest with constant velocity and acceleration in nonlinear formulation. Small-time expansion up to the third order of the surface elevation was found [4] and the hydrodynamic force on the cylinder was also discussed. The same problem was solved in [5] in the small-cylinder approximation. Waves excited in a two-layer fluid by a naturally rising sphere were considered by Mindlin [6].

In the present work, we solve the nonlinear unsteady problem using the method of reducing the initial Cauchy–Poisson problem to a system of integrodifferential equations on the free boundary. This method has been previously used in studying the well-posedness of problems of motion of a fluid with free boundaries [7]. Using this method, Makarenko [8] obtained and justified the dipole approximation for the problem of motion of a circular cylinder below the free surface. In [9, 10], the unique solvability (local in time) of the unsteady problem of motion of a submerged sphere was proved, and the method of its approximate modeling by a system of multipoles concentrated in the center was described.

The main objective of the present work is to construct the small-time expansion of the free surface elevation (with allowance for nonlinearity of boundary conditions) and the force acting on the sphere.

1. Formulation of the Problem. We consider the motion of a completely submerged sphere in an infinitely deep inviscid incompressible fluid with a free surface from above. The gravitational acceleration g is directed downward along the z axis, and the Ox_1x_2 plane coincides with the undisturbed free surface. A sphere of radius R moves with a constant acceleration $\boldsymbol{A}$. We choose $|\boldsymbol{A}|$ as a unit of acceleration and the distance H from the undisturbed free surface to the sphere center at the initial time as a unit of spatial variables. Then, the units of velocity, potential, time, and pressure are $\sqrt{|\boldsymbol{A}|H}$, $H\sqrt{|\boldsymbol{A}|H}$, $\sqrt{H/|\boldsymbol{A}|}$, and $\rho|\boldsymbol{A}|H$ (ρ is the constant density of the fluid), respectively. We introduce a dimensionless parameter — the Froude number $\mathrm{Fr} = \sqrt{|\boldsymbol{A}|/g}$. The main dimensionless parameters of the problem are $\varepsilon = R/H$, the coordinates of the sphere center $\boldsymbol{x}_c = (a_1t^2/2, a_2t^2/2, a_3t^2/2 - 1)$, time t, sphere velocity $\boldsymbol{v}_c = (a_1t, a_2t, a_3t)$, and acceleration $\boldsymbol{a} = (a_1, a_2, a_3)$, $|\boldsymbol{a}| = 1$. The equation for the free surface $\Gamma(t)$ has the form $z = h(x,t)$, $x = (x_1, x_2)$. The spatial variables x_1, x_2, and z here are dimensionless.

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The Cauchy–Poisson problem for the dimensionless velocity potential $\Phi(x_1, x_2, z, t)$ has the form

$$\Delta\Phi = 0 \qquad (-\infty < z < h(x,t), \quad |\boldsymbol{x} - \boldsymbol{x}_c| > \varepsilon),$$

$$h_t + \Phi_{x_1} h_{x_1} + \Phi_{x_2} h_{x_2} = \Phi_z, \qquad \Phi_t + |\nabla\Phi|^2/2 + \mathrm{Fr}^{-2} h = 0 \quad \text{on } \Gamma(t),$$

$$\boldsymbol{n} \cdot (\nabla\Phi(x,z,t) - \boldsymbol{v}_c(t)) = 0 \quad \text{on sphere } S_\varepsilon, \quad |\nabla\Phi| \to 0, \ h(x,t) \to 0 \quad \text{for } |x| + |z| \to \infty,$$

$$h(x,0) = h_0(x), \quad \Phi(x,z,0) = \Phi_0(x,z), \quad \Delta\Phi_0 = 0, \quad \boldsymbol{n} \cdot (\nabla\Phi_0(x,z) - \boldsymbol{v}_c(0)) = 0,$$

where $\boldsymbol{x} = (x, z)$ and $\boldsymbol{n}$ is the unit vector of the normal to the sphere S_ε.

2. Reduction to the Boundary. In the initial problem, the domain of definition of the sought function is unknown. By introducing the auxiliary functions $\varphi(x,t) = \Phi(x, h(x,t), t)$ and $\psi(x,t) = \Phi_z(x, h(x,t), t)$, we divide the Cauchy–Poisson problem into the Cauchy problem on the free surface and a mixed boundary-value problem for the Laplace equation in the domain occupied by the fluid. The Cauchy problem is formulated with the help of the conditions on the free surface [7]

$$h_t = -\nabla\varphi\nabla h + (1 + |\nabla h|^2)\psi, \qquad \varphi_t = -|\nabla\varphi|^2/2 - \mathrm{Fr}^{-2} h + (1 + |\nabla h|^2)\psi^2/2 \tag{2.1}$$

and initial conditions

$$h(x,0) = h_0(x), \qquad \varphi(x,0) = \Phi_0(x, h_0(x)).$$

System (2.1) is closed by an integral equation obtained using the Green formula

$$4\pi\Phi(\boldsymbol{x}) = \int_S G(\boldsymbol{x},\boldsymbol{y})\boldsymbol{n}\cdot\nabla\Phi(\boldsymbol{y})\,dS - \int_S \Phi(\boldsymbol{y})\boldsymbol{n}\cdot\nabla G(\boldsymbol{x},\boldsymbol{y})\,dS.$$

Here $\boldsymbol{y} = (y, h(y))$, $y = (y_1, y_2)$, $S = \Gamma(t) \cup S_\varepsilon$, $\boldsymbol{n}$ is the normal to the boundary S, and $G(\boldsymbol{x},\boldsymbol{y})$ is the Green function of the external Neumann problem for a sphere [11]:

$$G(\boldsymbol{x},\boldsymbol{y}) = 1/|\boldsymbol{x} - \boldsymbol{y}| + N(\boldsymbol{x},\boldsymbol{y}),$$

$$N(\boldsymbol{x},\boldsymbol{y}) = \frac{\varepsilon}{|\boldsymbol{x} - \boldsymbol{x}_c||(\boldsymbol{x} - \boldsymbol{x}_c)^* - (\boldsymbol{y} - \boldsymbol{x}_c)|} + \frac{1}{\varepsilon}\ln\frac{|\boldsymbol{x} - \boldsymbol{x}_c||\boldsymbol{y} - \boldsymbol{x}_c| - (\boldsymbol{x} - \boldsymbol{x}_c, \boldsymbol{y} - \boldsymbol{x}_c)}{\varepsilon^2 + |\boldsymbol{x} - \boldsymbol{x}_c||(\boldsymbol{x} - \boldsymbol{x}_c)^* - (\boldsymbol{y} - \boldsymbol{x}_c)| - (\boldsymbol{x} - \boldsymbol{x}_c, \boldsymbol{y} - \boldsymbol{x}_c)}$$

[the asterisk indicates inversion $(\boldsymbol{x} - \boldsymbol{x}_c)^* = \varepsilon^2(\boldsymbol{x} - \boldsymbol{x}_c)/|\boldsymbol{x} - \boldsymbol{x}_c|^2$ of the point $\boldsymbol{x}$ with respect to the sphere S_ε]. Since, by construction, we have $\boldsymbol{n}\cdot\nabla G(\boldsymbol{x},\boldsymbol{y})|_{S_\varepsilon} = 0$, one of the integrals over the sphere surface vanishes, and the other is the solution of the problem of sphere motion in an infinite fluid flow:

$$\frac{1}{4\pi}\int_{S_\varepsilon} G(\boldsymbol{x},\boldsymbol{y})\boldsymbol{n}\cdot\nabla\Phi(\boldsymbol{y})\,dS = \frac{\varepsilon^3(\boldsymbol{v}_c, \boldsymbol{x} - \boldsymbol{x}_c)}{2\,|\boldsymbol{x} - \boldsymbol{x}_c|^3}.$$

Then, integrals over the free surface only remain in the Green formula, i.e.,

$$4\pi\Phi(\boldsymbol{x}) = \int_{\Gamma(t)} G(\boldsymbol{x},\boldsymbol{y})\boldsymbol{n}\cdot\nabla\Phi(\boldsymbol{y})\,dS - \int_{\Gamma(t)} \Phi(\boldsymbol{y})\boldsymbol{n}\cdot\nabla G(\boldsymbol{x},\boldsymbol{y})\,dS - 2\pi\,\frac{\varepsilon^3(\boldsymbol{v}_c, \boldsymbol{x} - \boldsymbol{x}_c)}{|\boldsymbol{x} - \boldsymbol{x}_c|^3}.$$

We differentiate the resultant equality with respect to z and direct the point $\boldsymbol{x}$ to the free surface. Taking into account the jump of the potential of the double layer, equal to $-2\pi\Phi_z$, we obtain

$$2\pi\psi(x,t) = -\int_{R^2} \frac{h(x,t) - h(y,t)}{|\boldsymbol{x} - \boldsymbol{y}|^3}\,[(1 + |\nabla h|^2)\psi - \nabla h\nabla\varphi]\,dy_1\,dy_2$$

$$+ \int_{R^2} \frac{(x - y)\nabla\varphi(y,t)}{|\boldsymbol{x} - \boldsymbol{y}|^3}\,dy_1\,dy_2 - \frac{2\pi\varepsilon^3 a_3 t}{|\boldsymbol{x} - \boldsymbol{x}_c|^3} + 6\pi\,\frac{\varepsilon^3(h(x,t) - a_3 t^2/2 + 1)}{|\boldsymbol{x} - \boldsymbol{x}_c|^5}\,\boldsymbol{v}_c\cdot(\boldsymbol{x} - \boldsymbol{x}_c)$$

$$+ \int_{R^2} N_z(\boldsymbol{x},\boldsymbol{y})[(1 + |\nabla h|^2)\psi - \nabla h\nabla\varphi]\,dy_1\,dy_2 - \int_{R^2} \boldsymbol{\nu}\cdot\nabla N_z(\boldsymbol{x},\boldsymbol{y})\varphi(y,t)\,dy_1\,dy_2, \tag{2.2}$$

where $\boldsymbol{x} = (x, h(x,t))$ and $\boldsymbol{\nu} = (-h_{y_1}, -h_{y_2}, 1)$. In the right side of Eq. (2.2), the first two integrals do not depend explicitly on ε. They are also retained in (2.2) if the sphere radius equals zero, which yields an equation for the problem of free waves. The integrated terms correspond to a dipole moving in an infinite fluid. The last two integrals are responsible for the sphere–free surface interaction and vanish at $\varepsilon = 0$. Equations (2.1) and (2.2) form a closed system. Its solution allows one to reconstruct the velocity potential of the fluid everywhere in the flow region. Then, using the Cauchy–Lagrange integral, we find the pressure of the fluid on the moving sphere.

The function $N(\boldsymbol{x}, \boldsymbol{y})$ can be represented as a series in powers of ε:

$$N(\boldsymbol{x},\boldsymbol{y}) = \frac{\varepsilon}{|\boldsymbol{x}-\boldsymbol{x}_c(t)||\boldsymbol{y}-\boldsymbol{x}_c(t)|} + \frac{1}{\varepsilon}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\Big\{\sum_{n=1}^{\infty}\frac{\varepsilon^{2n}}{|\boldsymbol{x}-\boldsymbol{x}_c(t)|^n|\boldsymbol{y}-\boldsymbol{x}_c(t)|^n}\Big\}^k$$

$$+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{(n+1)!}\Big\{\frac{\varepsilon}{|\boldsymbol{x}-\boldsymbol{x}_c(t)|\,|\boldsymbol{y}-\boldsymbol{x}_c(t)|}\Big\}^{2n+1}\Big(\varepsilon^2 - 2(\boldsymbol{x}-\boldsymbol{x}_c(t),\boldsymbol{y}-\boldsymbol{x}_c(t))\Big)^n$$

$$+\frac{1}{\varepsilon}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\Big\{\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{(n+1)!}\Big(\frac{\varepsilon}{|\boldsymbol{x}-\boldsymbol{x}_c(t)|\,|\boldsymbol{y}-\boldsymbol{x}_c(t)|}\Big)^{n+1}$$

$$\times\Big(1-\frac{(\boldsymbol{x}-\boldsymbol{x}_c(t),\boldsymbol{y}-\boldsymbol{x}_c(t))}{|\boldsymbol{x}-\boldsymbol{x}_c(t)|\,|\boldsymbol{y}-\boldsymbol{x}_c(t)|}\Big)^n\Big(1-\frac{\varepsilon^2}{|\boldsymbol{x}-\boldsymbol{x}_c(t)|\,|\boldsymbol{y}-\boldsymbol{x}_c(t)|}\Big)^{-2n}\Big\}^k.$$

This series converges if $\varepsilon^2/|\boldsymbol{x}-\boldsymbol{x}_c(t)|\,|\boldsymbol{y}-\boldsymbol{x}_c(t)| < 1$, where the points $\boldsymbol{x}$ are $\boldsymbol{y}$ are located on the free surface. This condition means that the sphere during its motion should be completely submerged into the fluid. After transformations, we obtain

$$N(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\frac{\varepsilon^3(\boldsymbol{x}-\boldsymbol{x}_c,\boldsymbol{y}-\boldsymbol{x}_c)}{|\boldsymbol{x}-\boldsymbol{x}_c|^3|\boldsymbol{y}-\boldsymbol{x}_c|^3} - \frac{\varepsilon^5}{|\boldsymbol{x}-\boldsymbol{x}_c|^3|\boldsymbol{y}-\boldsymbol{x}_c|^3} + \sum_{n=3}^{\infty}\frac{\varepsilon^{2n-1}}{|\boldsymbol{x}-\boldsymbol{x}_c|^n|\boldsymbol{y}-\boldsymbol{x}_c|^n}\sum_{k=1}^{n}\frac{(-1)^k}{k}\frac{k!}{i_1!\ldots i_n!}$$

$$+\sum_{n=2}^{\infty}\sum_{k=0}^{n}(-1)^n\frac{(2n-1)!!}{(2n)!!}C_n^k(-2)^k\varepsilon^{4n-2k+1}\frac{(\boldsymbol{x}-\boldsymbol{x}_c,\boldsymbol{y}-\boldsymbol{x}_c)^k}{|\boldsymbol{x}-\boldsymbol{x}_c|^{2n+1}|\boldsymbol{y}-\boldsymbol{x}_c|^{2n+1}}$$

$$+\sum_{i=4}^{\infty}\sum_{k=1}^{[i/2]}\frac{(-1)^k}{k}\frac{k!}{j_2!\ldots j_i!}\sum_{m=0}^{\sum_{q=2}^{i}[\frac{q-1}{2}]j_q}B_{k,i,m,j_2,\ldots,j_i}\varepsilon^{2i-2m-1}\frac{(\boldsymbol{x}-\boldsymbol{x}_c,\boldsymbol{y}-\boldsymbol{x}_c)^m}{|\boldsymbol{x}-\boldsymbol{x}_c|^i|\boldsymbol{y}-\boldsymbol{x}_c|^i}$$

($i_1 + 2i_2 + \ldots + ni_n = n$, $i_1 + i_2 + \ldots + i_n = k$, $2j_2 + \ldots + ij_i = i$, and $j_2 + \ldots + j_i = k$). Here $B_{k,i,m,j_2,\ldots,j_i}$ are numerical coefficients

$$B_{k,i,m,j_2,\ldots,j_i} = \sum_{m_2=\max\left(0,m-\sum_{q=2,q\neq 2}^{i}[\frac{q-1}{2}]j_q\right)}^{\min(m,0)}\cdots\sum_{m_{i-1}=\max\left(0,m-\sum_{q=2,q\neq i-1}^{i}[\frac{q-1}{2}]j_q\right)}^{\min\left(m,[\frac{i-1}{2}]j_i\right)}A_{2,m_2,j_2}$$

$$\cdots A_{i-1,m_{i-1},j_{i-1}}A_{i,m-m_2-\ldots-m_{i-1},j_i};$$

$$A_{p,l_p,j_p} = C_{j_p}^{k_1}C_{j_p-k_1}^{k_2}\cdots C_{j_p-k_1-\ldots-k_{[(p-1)/2]-1}}^{k_{[(p-1)/2]}}a_{p,0}^{j_p-(k_1+\ldots+k_{[(p-1)/2]})}a_{p,1}^{k_1}\cdots a_{p,[(p-1)/2]}^{k_{[(p-1)/2]}};$$

$$k_1 + 2k_2 + \ldots + [(p-1)/2]k_{[(p-1)/2]} = l_p, \qquad 0 \leqslant l_p \leqslant [(p-1)/2]j_p;$$

$$k_1 + k_2 + \ldots + k_{[(p-1)/2]} \leqslant j_p;$$

$$a_{p,j} = (-1)^j\sum_{\max(1,j)}^{p-j-1}(-1)^n\frac{(2n-1)!!}{(n+1)!}C_{n+p-j-2}^{p-j-n-1}C_n^j;$$

$i_1, \ldots, i_n$, $j_2, \ldots, j_i$, and $k_1, \ldots, k_{[(p-1)/2]}$ are integer nonnegative numbers, and $[\nu]$ is the integer part of the number ν.

Substituting the resultant representation of the function $N(\boldsymbol{x},\boldsymbol{y})$ in the form of a series in powers of ε into the Green formula for the velocity potential, we may draw the following conclusion. Exact satisfaction of the boundary condition on the sphere surface corresponds to the assumption that an infinite number of multipoles are concentrated in the center of the sphere; the strength of the multipoles depends on the parameter ε, instantaneous elevation of the free surface, and velocity potential of the fluid on it. The small value of the parameter ε implies that the sphere is initially located rather far from the free surface. We seek the solution of the initial problem in two approximate formulations. The first approximation is obtained by discarding terms of order ε^5 and higher in the kernel $N(\boldsymbol{x},\boldsymbol{y})$ of the integral equation (2.2); the second approximation is obtained by discarding terms of order ε^7 and higher. The first and second approximations are called dipole and modified dipole approximations, respectively

3. Derivation of Auxiliary Relations. Here, we derive relations valid for all approximations considered. At the initial time, the fluid is at rest, and the initial conditions are $h(x,0)=0$ and $\varphi(x,0)=0$. The solution of system (2.1), (2.2) is sought in the form of series in powers of t:

$$(h,\varphi,\psi)=(0,0,\psi_0)+t(h_1,\varphi_1,\psi_1)+t^2(h_2,\varphi_2,\psi_2)/2+t^3(h_3,\varphi_3,\psi_3)/3!+\ldots\ .$$

Equations (2.1) yield recurrent formulas relating the expansion coefficients of unknown functions:

$$h_1=\psi_0,\quad h_2=\psi_1,\quad \varphi_1=\psi_0^2/2,\quad \varphi_2=-\mathrm{Fr}^{-2}\psi_0+\psi_0\psi_1;$$

for $i\geqslant 2$, we obtain

$$h_{i+2}=-\sum_{k=1}^{i-1}\nabla\varphi_k\nabla h_{i-k}+\psi_i+\sum_{k=2}^{i}\psi_{i-k}\sum_{n=1}^{k-1}\nabla h_n\nabla h_{k-n},$$

$$\varphi_{i+1}=-\frac{1}{2}\sum_{k=1}^{i-1}\nabla\varphi_k\nabla\varphi_{i-k}-\mathrm{Fr}^{-2}h_i+\frac{1}{2}\sum_{k=0}^{i}\psi_k\psi_{i-k}+\frac{1}{2}\sum_{k=2}^{i}\sum_{n=1}^{k-1}\nabla h_n\nabla h_{k-n}\sum_{m=1}^{i-k}\psi_m\psi_{i-k-m}.$$

It is clear that the coefficients of expansion of the free surface elevation and velocity potential of the fluid in powers of t are expressed via the expansion coefficients of the function ψ. The equations for ψ_i are derived by expanding Eq. (2.2) in powers of t.

We introduce the following notation for integral operators:

$$A_0\psi=-\frac{1}{2\pi}\int\limits_{R^2}\frac{h(x,t)-h(y,t)}{|\boldsymbol{x}-\boldsymbol{y}|^3}(1+|\nabla h(y,t)|^2)\psi(y,t)\,dy_1\,dy_2,$$

$$B_0\varphi=\frac{1}{2\pi}\int\limits_{R^2}\frac{(x-y)+(h(x,t)-h(y,t))\nabla h(y,t)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,\nabla\varphi(y,t)\,dy_1\,dy_2,$$

$$A_\varepsilon\psi=\frac{1}{2\pi}\int\limits_{R^2}N_z(\boldsymbol{x},\boldsymbol{y})(1+|\nabla h(y,t)|^2)\psi(y,t)\,dy_1\,dy_2,$$

$$B_\varepsilon\varphi=-\frac{1}{2\pi}\int\limits_{R^2}N_z(\boldsymbol{x},\boldsymbol{y})\nabla h(y,t)\nabla\varphi(y,t)\,dy_1\,dy_2-\frac{1}{2\pi}\int\limits_{R^2}\boldsymbol{\nu}\cdot\nabla N_z(\boldsymbol{x},\boldsymbol{y})\varphi(y,t)\,dy_1\,dy_2.$$

We designate the integrated terms of Eq. (2.2) as

$$w_{\text{dip}}=-\frac{\varepsilon^3a_3t}{|\boldsymbol{x}-\boldsymbol{x}_c|^3}+3\,\frac{\varepsilon^3(h(x,t)-a_3t^2/2+1)}{|\boldsymbol{x}-\boldsymbol{x}_c|^5}\,\boldsymbol{v}_c\cdot(\boldsymbol{x}-\boldsymbol{x}_c).$$

We rewrite the equation for the normal derivative using the new notation:

$$\psi=A_0\psi+B_0\varphi+w_{\text{dip}}+A_\varepsilon\psi+B_\varepsilon\varphi. \tag{3.1}$$

Note, the function $N(\boldsymbol{x},\boldsymbol{y})$ has the third order of smallness in ε; therefore, the operators A_ε and B_ε are quantities of the same order. With allowance for expansion of h in powers of t, the integral operators can be represented in the form of the series

$$A_0=tA_0^{(1)}+\frac{t^2}{2}\,A_0^{(2)}+\frac{t^3}{6}\,A_0^{(3)}+\frac{t^4}{24}\,A_0^{(4)}+\ldots,\qquad B_0=B_0^{(0)}+\frac{t^2}{2}\,B_0^{(2)}+\frac{t^4}{24}\,B_0^{(4)}+\frac{t^5}{120}\,B_0^{(5)}+\ldots,$$

$$A_\varepsilon=A_\varepsilon^{(0)}+\frac{t^2}{2}\,A_\varepsilon^{(2)}+\frac{t^4}{24}\,A_\varepsilon^{(4)}+\frac{t^5}{120}\,A_\varepsilon^{(5)}+\ldots,\qquad B_\varepsilon=B_\varepsilon^{(0)}+\frac{t^2}{2}\,B_\varepsilon^{(2)}+\frac{t^3}{6}\,B_\varepsilon^{(3)}+\ldots\ .$$

The absence of terms of order t and t^3 in the above formulas is explained by the quadratic dependence of the coordinates of the sphere center on time. The expansion of the function w_{dip} contains only odd powers of t, and the operators with the subscript ε are determined by the kernel $N(\boldsymbol{x}, \boldsymbol{y})$ and are different in different approximations.

The functions ψ_i satisfy the integral equations obtained from Eq. (3.1) by equating coefficients at identical powers of t:

$$(I - A_\varepsilon^{(0)})\psi_i = f_i \qquad (i = 0, 1, 2, \ldots). \tag{3.2}$$

The right sides f_i have the form

$$f_0 = 0, \quad f_1 = w_{\mathrm{dip}}^{(1)} + A_0^{(1)}\psi_0, \quad f_2 = A_0^{(2)}\psi_0 + A_\varepsilon^{(2)}\psi_0 + 2A_0^{(1)}\psi_1,$$

$$f_3 = w_{\mathrm{dip}}^{(3)} + B_0^{(0)}\varphi_3 + B_\varepsilon^{(0)}\varphi_3 + 3A_0^{(1)}\psi_2 + A_0^{(3)}\psi_0 + 3A_0^{(2)}\psi_1 + 3A_\varepsilon^{(2)}\psi_1,$$

$$f_4 = B_0^{(0)}\varphi_4 + B_\varepsilon^{(0)}\varphi_4 + 4A_0^{(1)}\psi_3 + 6A_0^{(2)}\psi_2 + 4A_0^{(3)}\psi_1 + 6A_\varepsilon^{(2)}\psi_2 + A_\varepsilon^{(3)}\psi_1 + A_\varepsilon^{(4)}\psi_0,$$

$$f_5 = w_{\mathrm{dip}}^{(5)} + B_0^{(0)}\varphi_5 + 10B_0^{(2)}\varphi_3 + B_\varepsilon^{(0)}\varphi_5 + 10B_\varepsilon^{(2)}\varphi_3 + 5A_0^{(1)}\psi_4 + 10A_0^{(2)}\psi_3 + 10A_0^{(3)}\psi_2 + 5A_0^{(4)}\psi_1$$

$$+ 10A_\varepsilon^{(2)}\psi_3 + 5A_\varepsilon^{(4)}\psi_1 + A_0^{(5)}\psi_0 + A_\varepsilon^{(5)}\psi_0, \ \ldots .$$

4. Dipole Approximation. In this subsection, we seek the solution of system (2.1), (2.2) with accuracy to ε^5. This is called the dipole approximation, since the sphere is modeled by a dipole moving with a velocity $\boldsymbol{v}_c$ in an infinite fluid. In the dipole approximation, we have

$$N(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\,\frac{\varepsilon^3(\boldsymbol{x} - \boldsymbol{x}_c(t), \boldsymbol{y} - \boldsymbol{x}_c(t))}{|\boldsymbol{x} - \boldsymbol{x}_c(t)|^3 |\boldsymbol{y} - \boldsymbol{x}_c(t)|^3}.$$

In this case, the operator $A_\varepsilon^{(0)}\psi_i$ has the form

$$A_\varepsilon^{(0)}\psi_i = \frac{1}{4\pi}\,\frac{\varepsilon^3}{(x^2+1)^{3/2}} \int\limits_{R^2} \frac{\psi_i(y)}{(y^2+1)^{3/2}}\,dy_1\,dy_2 - \frac{3}{4\pi}\,\frac{\varepsilon^3 x_1}{(x^2+1)^{5/2}} \int\limits_{R^2} \frac{y_1\psi_i(y)}{(y^2+1)^{3/2}}\,dy_1\,dy_2$$

$$- \frac{3}{4\pi}\,\frac{\varepsilon^3 x_2}{(x^2+1)^{5/2}} \int\limits_{R^2} \frac{y_2\psi_i(y)}{(y^2+1)^{3/2}}\,dy_1\,dy_2 - \frac{3}{4\pi}\,\frac{\varepsilon^3}{(x^2+1)^{5/2}} \int\limits_{R^2} \frac{\psi_i(y)}{(y^2+1)^{3/2}}\,dy_1\,dy_2. \tag{4.1}$$

The convergence of the Neumann series for the operator $A_\varepsilon^{(0)}$ for small ε was established in [12]. Hence, if the right side of the integral equation (3.2), where the operator $A_\varepsilon^{(0)}\psi_i$ is determined by formula (4.1), equals zero, then the only solution of this equation also equals zero. Since $\psi_0 = 0$, the operator is $A_0^{(1)} = 0$, and the right side of the equation for ψ_2 is also equal to zero. Then, we have $\psi_2 = 0$; hence, $A_0^{(3)} = 0$ and $\psi_4 = 0$.

The right side of Eq. (3.2) for ψ_1 remains unchanged for all approximations in ε:

$$f_1 = -\frac{\varepsilon^3 a_3}{(x^2+1)^{3/2}} + \frac{3\varepsilon^3(a_1x_1 + a_2x_2 + a_3)}{(x^2+1)^{5/2}}.$$

Discarding terms of order ε^6 in the solution of the integral equation for ψ_1, we obtain

$$\psi_1(x) = -\frac{\varepsilon^3 a_3}{(x^2+1)^{3/2}} + \frac{3\varepsilon^3(a_1x_1 + a_2x_2 + a_3)}{(x^2+1)^{5/2}}.$$

The solution for ψ_1, as the right side of Eq. (3.2), has the third order of smallness in terms of ε. Thus, the operator $A_\varepsilon^{(0)}\psi_1$ is of order ε^6, and finally, the solution of Eq. (3.2), within the accepted accuracy, is equal to its right side. In the dipole approximation, this is valid for all ψ_i.

The right side of Eq. (3.2) for ψ_3 is determined by the formula

$$f_3 = w_{\mathrm{dip}}^{(3)} + B_0^{(0)}\varphi_3,$$

since $A_0^{(1)} = A_0^{(3)} = 0$, and the remaining terms in f_3 are discarded because they have the sixth order of smallness in terms of ε. Thus, we obtain

$$f_3 = -\frac{9\varepsilon^3}{(x^2+1)^{5/2}} - 18\varepsilon^3 a_3 \frac{a_1x_1 + a_2x_2 + a_3}{(x^2+1)^{5/2}}$$

$$+ 45\varepsilon^3 \frac{(a_1x_1 + a_2x_2 + a_3)^2}{(x^2+1)^{7/2}} + \frac{1}{2\pi} \int\limits_{R^2} \frac{(x-y)\cdot\nabla\varphi_3(y)}{|x-y|^3}\, dy_1\, dy_2.$$

The integral in the last formula arises because of the operator $B_0^{(0)}\varphi_3$. In this approximation, the nonlinear terms in the free-surface conditions do not contribute to the solution. Therefore, we have

$$\varphi_3(x) = \mathrm{Fr}^{-2} \frac{\varepsilon^3 a_3}{(x^2+1)^{3/2}} - \mathrm{Fr}^{-2} \frac{3\varepsilon^3(a_1x_1 + a_2x_2 + a_3)}{(x^2+1)^{5/2}}.$$

To calculate $B_0^{(0)}\varphi_3$, we use the Fourier transform for the Riesz integral operator

$$Ru(x) = \frac{1}{2\pi} \int\limits_{R^2} \frac{(x-y)u(y)}{|x-y|^3}\, dy_1\, dy_2 \quad [u = (u_1, u_2)],$$

for which the following formula is valid: $\widehat{Ru}(\xi) = i\,\mathrm{sign}\,\xi\,\hat{u}(\xi)$. Here $\xi = (\xi_1, \xi_2)$ and $\hat{u}(\xi) = (\hat{u_1}(\xi), \hat{u_1}(\xi))$; the vector is $\mathrm{sign}\,\xi = 0$ for $\xi = 0$ or $\mathrm{sign}\,\xi = \xi/|\xi|$ otherwise. We assume that $u_1 = \varphi_{3y_1}$ and $u_2 = \varphi_{3y_2}$. The Fourier transform of the function $B_0^{(0)}\varphi_3$ has the form

$$\widehat{B_0^{(0)}\varphi_3}(\xi) = \mathrm{Fr}^{-2}\varepsilon^3 \mathrm{e}^{-|\xi|}(ia_3|\xi|^2 - a_1|\xi|\xi_1 - a_2|\xi|\xi_2).$$

Using the inverse Fourier transform, we obtain

$$\frac{1}{2\pi} \int\limits_{R^2} \frac{(x-y)\cdot\nabla\varphi_3(y)}{|x-y|^3}\, dy_1\, dy_2 = 3\mathrm{Fr}^{-2}\varepsilon^3 \frac{a_1x_1 + a_2x_2 + 3a_3}{(x^2+1)^{5/2}} - 15\mathrm{Fr}^{-2}\varepsilon^3 \frac{a_1x_1 + a_2x_2 + a_3}{(x^2+1)^{7/2}}.$$

Then, we have

$$\psi_3(x) = 3\mathrm{Fr}^{-2}\,\varepsilon^3 \frac{a_1x_1 + a_2x_2 + 3a_3}{(x^2+1)^{5/2}} - 15\mathrm{Fr}^{-2}\,\varepsilon^3 \frac{a_1x_1 + a_2x_2 + a_3}{(x^2+1)^{7/2}}$$

$$- \frac{9\varepsilon^3}{(x^2+1)^{5/2}} - 18\varepsilon^3 a_3 \frac{a_1x_1 + a_2x_2 + a_3}{(x^2+1)^{5/2}} + 45\varepsilon^3 \frac{(a_1x_1 + a_2x_2 + a_3)^2}{(x^2+1)^{7/2}}.$$

As a result, we obtain the free-surface elevation in the dipole approximation

$$h(x,t) = t^2\,\frac{3}{2}\,\varepsilon^3 \frac{a_1x_1 + a_2x_2 + a_3(2-x^2)}{(x^2+1)^{5/2}} - t^4\,\frac{3}{8}\,\varepsilon^3 \frac{1}{(x^2+1)^{5/2}}$$

$$- t^4\,\frac{3}{4}\,\varepsilon^3\, a_3 \frac{a_1x_1 + a_2x_2 + a_3}{(x^2+1)^{5/2}} + t^4\,\frac{15}{8}\,\varepsilon^3 \frac{(a_1x_1 + a_2x_2 + a_3)^2}{(x^2+1)^{7/2}}$$

$$+\mathrm{Fr}^{-2}\,t^4\,\frac{1}{8}\,\varepsilon^3 \frac{a_1x_1 + a_2x_2 + 3a_3}{(x^2+1)^{5/2}} - \mathrm{Fr}^{-2}\,t^4\,\frac{5}{8}\,\varepsilon^3 \frac{a_1x_1 + a_2x_2 + a_3}{(x^2+1)^{7/2}} + O(t^6).$$

Figure 1 shows the surface of the fluid in the case of horizontal motion of the sphere along the x_1 axis. The sphere of radius $\varepsilon = 0.5$ moves with an acceleration $|\boldsymbol{A}| = g/4$. At the initial time, the free surface is the $z = 0$ plane. The motion of the sphere is considered until the time $t = 1$; per dimensionless time unit, the sphere passes a distance equal to half of its initial depth H. For the slow acceleration of the sphere considered, two first ridges of the wave wake are formed during the time period mentioned.

5. Modified Dipole Approximation. To obtain a system of equations of the modified dipole approximation, we make the following substitution in (2.2):

$$N(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2} \frac{\varepsilon^3(\boldsymbol{x} - \boldsymbol{x}_c(t), \boldsymbol{y} - \boldsymbol{x}_c(t))}{|\boldsymbol{x} - \boldsymbol{x}_c(t)|^3|\boldsymbol{y} - \boldsymbol{x}_c(t)|^3} - \frac{1}{3} \frac{\varepsilon^5}{|\boldsymbol{x} - \boldsymbol{x}_c(t)|^3|\boldsymbol{y} - \boldsymbol{x}_c(t)|^3} + \frac{\varepsilon^5(\boldsymbol{x} - \boldsymbol{x}_c(t), \boldsymbol{y} - \boldsymbol{x}_c(t))^2}{|\boldsymbol{x} - \boldsymbol{x}_c(t)|^5|\boldsymbol{y} - \boldsymbol{x}_c(t)|^5}.$$

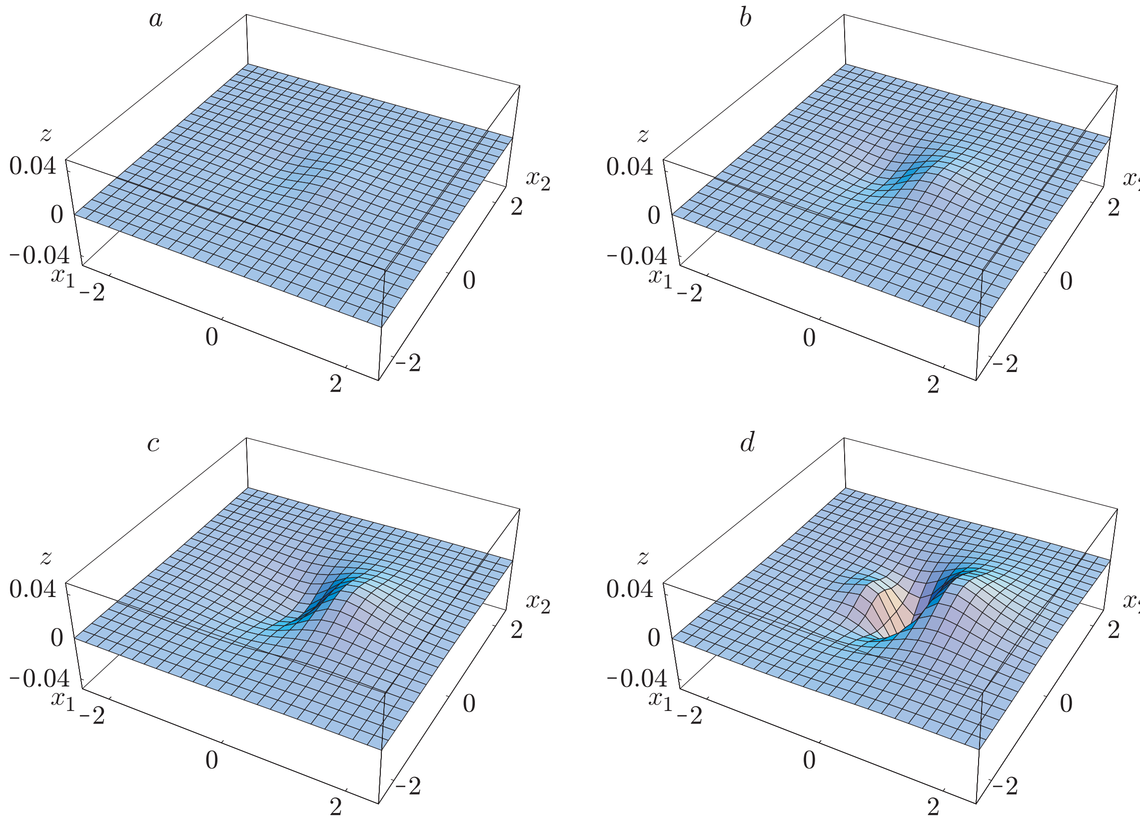

Fig. 1. Free surface in the case of horizontal motion of the sphere at different dimensionless times: $t = 0.25$ (a), 0.5 (b), 0.75 (c), and 1.0 (d).

In this case, in the solution, we neglect terms of the seventh and higher orders in ε. In this approximation, the operator $A_\varepsilon^{(0)}\psi_i$ has the form

$$A_\varepsilon^{(0)}\psi_i = \frac{1}{4\pi}\,\frac{\varepsilon^3}{(x^2+1)^{3/2}}\int\limits_{R^2}\frac{\psi_i(y)}{(y^2+1)^{3/2}}\,dy_1\,dy_2 - \frac{3}{4\pi}\,\frac{\varepsilon^3}{(x^2+1)^{5/2}}\int\limits_{R^2}\frac{(1+x\cdot y)\psi_i(y)}{(y^2+1)^{3/2}}\,dy_1\,dy_2$$

$$-\frac{1}{\pi}\,\frac{\varepsilon^5}{(x^2+1)^{5/2}}\int\limits_{R^2}\frac{(1+x\cdot y)\psi_i}{(y^2+1)^{5/2}}\,dy_1\,dy_2 + \frac{5}{2\pi}\,\frac{\varepsilon^5}{(x^2+1)^{7/2}}\int\limits_{R^2}\frac{(1+x\cdot y)^2\psi_i}{(y^2+1)^{5/2}}\,dy_1\,dy_2$$

$$-\frac{1}{2\pi}\,\frac{\varepsilon^5}{(x^2+1)^{5/2}}\int\limits_{R^2}\frac{\psi_i(y)}{(y^2+1)^{3/2}}\,dy_1\,dy_2.$$

As in the case of the dipole approximation, if the right side of Eq. (3.2) equals zero, then its only solution is also zero. Therefore, we have $\psi_0 = \psi_2 = \psi_4 = 0$ and, similar to the previous approximation, $A_0^{(1)} = A_0^{(3)} = 0$. The function

$$\psi_1(x) = -\frac{\varepsilon^3 a_3}{(x^2+1)^{3/2}} + \frac{3\varepsilon^3(a_1x_1 + a_2x_2 + a_3)}{(x^2+1)^{5/2}} - \frac{3}{16}\,\varepsilon^6\,\frac{a_1x_1 + a_2x_2 + 2a_3}{(x^2+1)^{5/2}} + \frac{1}{8}\,\varepsilon^6\,\frac{a_3}{(x^2+1)^{3/2}}$$

is the solution of Eq. (3.2) for ψ_1 in the modified dipole approximation. In this solution, we discarded terms of the eighth and higher orders in ε. Terms of the sixth order in ε in $\psi_1(x)$ appear because of the operator $A_\varepsilon^{(0)}\psi_i$. Thus, in this approximation, the sphere is simulated by a dipole whose strength is a sum of two terms. The first term if the strength of the classical dipole, and the second term is the strength of the induced dipole, which characterizes the interaction of the sphere and the free surface. Since the initial conditions of the problem are zero, the solution has the order ε^3. Being a functional of the instantaneous elevation of the free surface and velocity potential on the latter, the strength of the induced dipole is a quantity of order ε^6.

The main difficulties in solving the equation for the third coefficient in the expansion of the function ψ arise because of the presence of the terms $B_0^{(0)}\varphi_3$ and $A_0^{(2)}\psi_1$ in the right side of f_3. The modified dipole approximation takes into account the nonlinear terms of the conditions on the free surface (quantities without the factor Fr^{-2}):

$$\varphi_3(x) = \mathrm{Fr}^{-2}\,\frac{\varepsilon^3 a_3}{(x^2+1)^{3/2}} - 3\mathrm{Fr}^{-2}\varepsilon^3\,\frac{a_1x_1+a_2x_2+a_3}{(x^2+1)^{5/2}} + \mathrm{Fr}^{-2}\,\frac{3}{16}\,\varepsilon^6\,\frac{a_1x_1+a_2x_2+2a_3}{(x^2+1)^{5/2}}$$

$$-\,\mathrm{Fr}^{-2}\,\frac{1}{8}\,\varepsilon^6\,\frac{a_3}{(x^2+1)^{3/2}} + \Big(-\frac{\varepsilon^3 a_3}{(x^2+1)^{3/2}} + 3\varepsilon^3\,\frac{a_1x_1+a_2x_2+a_3}{(x^2+1)^{5/2}}\Big)^2.$$

The Fourier transform of the integral

$$B_0^{(0)}\varphi_3 = \frac{1}{2\pi}\int\limits_{R^2}\frac{(x-y)\cdot\nabla\varphi_3(y)}{((x_1-y_1)^2+(x_2-y_2)^2)^{3/2}}\,dy_1\,dy_2,$$

as in the dipole approximation, is calculated using the Fourier transform of the Riesz operator and has the form

$$\widehat{B_0^{(0)}\varphi_3} = \mathrm{Fr}^{-2}\varepsilon^3\mathrm{e}^{-|\xi|}(ia_3|\xi|^2 - a_1|\xi|\xi_1 - a_2|\xi|\xi_2)$$

$$+\,\mathrm{Fr}^{-2}\varepsilon^6\mathrm{e}^{-|\xi|}(ia_1|\xi|\xi_1 + ia_2|\xi|\xi_2 - a_3|\xi| + a_3|\xi|^2)/16 + i(\xi_1/|\xi|)\widehat{\psi^2_{1x_1}}(\xi) + i(\xi_2/|\xi|)\widehat{\psi^2_{1x_2}}(\xi).$$

The Fourier transform of the term that appears because of the nonlinearity of the dynamic boundary condition, has the form

$$i\,\frac{\xi_1}{|\xi|}\,\widehat{\psi^2_{1x_1}}(\xi) + i\,\frac{\xi_2}{|\xi|}\,\widehat{\psi^2_{1x_2}}(\xi) = \varepsilon^6\Big(\frac{1}{8}\,a_3^2|\xi|^3K_2(|\xi|) - \frac{1}{8}\,ia_3(a_1\xi_1+a_2\xi_2)|\xi|^3K_2(|\xi|)$$

$$-\,\frac{1}{8}\,a_3^2|\xi|^4K_3(|\xi|) + \frac{3}{64\cdot 2}\,|\xi|^4K_3(|\xi|) - \frac{3}{64\cdot 2}\,a_3^2|\xi|^4K_3(|\xi|) + \frac{3}{64\cdot 2}\,a_3^2|\xi|^5K_4(|\xi|)$$

$$-\,\frac{3}{64\cdot 2}(a_1\xi_1+a_2\xi_2)^2|\xi|^3K_2(|\xi|) - \frac{6}{64\cdot 2}\,a_1a_2|\xi|^3\xi_1\xi_2K_2(|\xi|) + \frac{3}{64}\,ia_3(a_1\xi_1+a_2\xi_2)|\xi|^4K_3(|\xi|)\Big).$$

Here $K_i(z)$ are modified Bessel functions of the second kind of order i (or Macdonald cylindrical functions). The result of action of the operator

$$A_0^{(2)}\psi_1 = -\frac{1}{2\pi}\int\limits_{R^2}\frac{h_2(x)-h_2(y)}{((x_1-y_1)^2+(x_2-y_2)^2)^{3/2}}\,\psi_1(y)\,dy_1\,dy_2$$

is also found using the Fourier transform:

$$\widehat{A_0^{(2)}\psi_1} = -\frac{1}{2\pi}\,|\xi|\int\limits_{R^2}\hat{\psi}_1(\xi-\eta)\hat{\psi}_1(\eta)\,d\eta_1\,d\eta_2 + \frac{1}{2\pi}\int\limits_{R^2}|\xi-\eta|\hat{\psi}_1(\xi-\eta)\hat{\psi}_1(\eta)\,d\eta_1\,d\eta_2.$$

Taking into account that the convolution of the Fourier transforms of two functions yields the Fourier transform for the product of these functions, we obtain

$$\widehat{A_0^{(2)}\psi_1} = -|\xi|\widehat{\psi_1^2}(\xi) + \hat{g}(\xi),$$

where

$$g(x) = 3\varepsilon^6a_3\,\frac{a_1x_1+a_2x_2+3a_3}{(x^2+1)^4} - 33\varepsilon^6a_3\,\frac{a_1x_1+a_2x_2+a_3}{(x^2+1)^5} - 9\varepsilon^6\,\frac{(a_1x_1+a_2x_2+a_3)^2}{(x^2+1)^5} + 45\varepsilon^6\,\frac{(a_1x_1+a_2x_2+a_3)^2}{(x^2+1)^6}.$$

The calculations yield

$$|\xi|\widehat{\psi_1^2}(\xi) = -i(\xi_1/|\xi|)\widehat{\psi^2_{1x_1}}(\xi) - i(\xi_2/|\xi|)\widehat{\psi^2_{1x_2}}(\xi).$$

To find the inverse Fourier transform for the functions $\widehat{B_0^{(0)}\varphi_3}$ and $\widehat{A_0^{(2)}\psi_1}$, we use the integral formula (see formula 6.576.3 in [13])

$$F((\nu-\lambda+\mu+1)/2, (\nu-\lambda-\mu+1)/2; \nu+1; -a^2)$$

$$= \frac{2^{\lambda+1}a^{-\nu}\Gamma(1+\nu)}{\Gamma((\nu-\lambda+\mu+1)/2)\Gamma((\nu-\lambda-\mu+1)/2)}\int\limits_0^\infty x^{-\lambda}K_\mu(x)J_\nu(ax)\,dx,$$

where $\nu - \lambda + 1 > |\mu|$, $\Gamma(z)$ is the gamma-function, and $J_\nu(z)$ is the Bessel function. Because of the presence of the Macdonald functions in the expressions for $\widehat{B_0^{(0)}\varphi_3}$ and $\widehat{A_0^{(2)}\psi_1}$, the right side of the equation for ψ_3 and, hence, the solution itself are expressed via hypergeometric functions.

Thus, we obtain the time evolution of the free surface in the modified dipole approximation with an accuracy to t^6:

$$h(x,t) = t^2\Big(\frac{3}{2}\,\varepsilon^3\,\frac{a_1x_1 + a_2x_2 + a_3(2 - x^2)}{(x^2+1)^{5/2}} - \frac{3}{32}\,\varepsilon^6\,\frac{a_1x_1 + a_2x_2 + 2a_3}{(x^2+1)^{5/2}} + \frac{1}{16}\,\varepsilon^6\,\frac{a_3}{(x^2+1)^{3/2}}\Big)$$

$$- t^4\,\frac{3}{8}\,\varepsilon^3\,\frac{1}{(x^2+1)^{5/2}} - t^4\,\frac{3}{4}\,\varepsilon^3\,a_3\,\frac{a_1x_1 + a_2x_2 + a_3}{(x^2+1)^{5/2}} + t^4\,\frac{15}{8}\,\varepsilon^3\,\frac{(a_1x_1 + a_2x_2 + a_3)^2}{(x^2+1)^{7/2}}$$

$$+ \mathrm{Fr}^{-2}t^4\,\frac{1}{8}\,\varepsilon^3\,\frac{a_1x_1 + a_2x_2 + 3a_3}{(x^2+1)^{5/2}} - \mathrm{Fr}^{-2}t^4\,\frac{5}{8}\,\varepsilon^3\,\frac{a_1x_1 + a_2x_2 + a_3}{(x^2+1)^{7/2}} + \frac{t^4}{24}\,\varepsilon^6\,\frac{18a_3^2 - 7\mathrm{Fr}^{-2}a_3}{16(x^2+1)^{3/2}}$$

$$+ \frac{t^4}{24}\,\varepsilon^6\,\frac{9(1 - a_3^2)}{16(x^2+1)^{5/2}} + \frac{t^4}{8}\,\varepsilon^6\mathrm{Fr}^{-2}\,\frac{a_1x_1 + a_2x_2 + a_3}{8(x^2+1)^{5/2}} - \frac{t^4}{24}\,\varepsilon^6\,\frac{45(a_1x_1 + a_2x_2 + a_3)^2}{16(x^2+1)^{7/2}}$$

$$-\frac{t^4}{24}\,\varepsilon^6\,\frac{45a_3(a_1x_1 + a_2x_2 + a_3)}{16(x^2+1)^{7/2}} + \frac{t^4}{24}\,\mathrm{Fr}^{-2}\varepsilon^6\,\frac{15(a_1x_1 + a_2x_2 + a_3)}{16(x^2+1)^{7/2}}$$

$$+ \frac{t^4}{24}\,\varepsilon^6\pi\Big\{ - a_3^2\,\frac{15}{8}\,F\Big(\frac{3}{2}, \frac{7}{2}; 1; -x^2\Big) + \frac{105}{128}\,(19a_3^2 - 3)F\Big(\frac{3}{2}, \frac{9}{2}; 1; -x^2\Big)$$

$$- \frac{3}{64}\,a_3^2\,\frac{945}{2}\,F\Big(\frac{3}{2}, \frac{11}{2}; 1; -x^2\Big) + \frac{315}{16}\,a_3(a_1x_1 + a_2x_2)F\Big(\frac{5}{2}, \frac{9}{2}; 1; -x^2\Big)$$

$$+ \frac{14{,}175}{128}\,a_3(a_1x_1 + a_2x_2)x^2F\Big(\frac{7}{2}, \frac{11}{2}; 3; -x^2\Big) - \frac{3}{64}(a_1x_1 + a_2x_2)^2\,\frac{14{,}175}{16}\,F\Big(\frac{7}{2}, \frac{11}{2}; 1; -x^2\Big)$$

$$- \frac{1}{128}(a_1x_1 + a_2x_2)^2x^2\,\frac{1{,}091{,}475}{16}\,F\Big(\frac{9}{2}, \frac{13}{2}; 3; -x^2\Big)$$

$$- \frac{1}{64}\,\frac{(a_1x_1 + a_2x_2)^2}{8}\,x^4\,\frac{42{,}567{,}525}{256}\,F\Big(\frac{11}{2}, \frac{15}{2}; 5; -x^2\Big)$$

$$- \frac{3}{64}\,a_3(a_1x_1 + a_2x_2)\,\frac{2835}{2}\,F\Big(\frac{5}{2}, \frac{11}{2}; 1; -x^2\Big)$$

$$- \frac{3}{64}\,a_3(a_1x_1 + a_2x_2)x^2\,\frac{155{,}925}{16}\,F\Big(\frac{7}{2}, \frac{13}{2}; 3; -x^2\Big) + \frac{3}{128}\,(1 - a_3^2)\,\frac{315}{2}\,F\Big(\frac{5}{2}, \frac{9}{2}; 1; -x^2\Big)$$

$$+ \frac{3}{128}\,(1 - a_3^2)x^2\,\frac{14{,}175}{16}\,F\Big(\frac{7}{2}, \frac{11}{2}; 3; -x^2\Big)\Big\} + O(t^6).$$

The expressions for the function $h(x,t)$ obtained in the dipole and modified dipole approximations allows us to compare the influence of the modified dipole and nonlinearity of the boundary conditions on the free-surface elevation of the fluid.

Figure 2 shows the free-surface elevation for both approximations at the time when the sphere has passed a distance equal to half of its initial depth. In the second approximation, the solutions of the problem with linear and nonlinear conditions on the free surface are given. The sphere radius is $\varepsilon = 0.6$, and the acceleration is $|\boldsymbol{A}| = g$. In the case of vertical motion of the sphere, the greatest difference in the solutions for this value of ε is observed; the difference is greater in the case of ascent than in the case of descent. In the case of horizontal motion of the sphere, the difference in the solutions is minimum.

6. Force Acting on the Sphere. We calculate the force acting on the sphere in the dipole approximation. The pressure in the fluid is found using the Cauchy–Lagrange integral:

$$p(x,z,t) = -\Phi_t(x,z,t) - |\nabla\Phi(x,z,t)|^2/2 - \mathrm{Fr}^{-2}z.$$

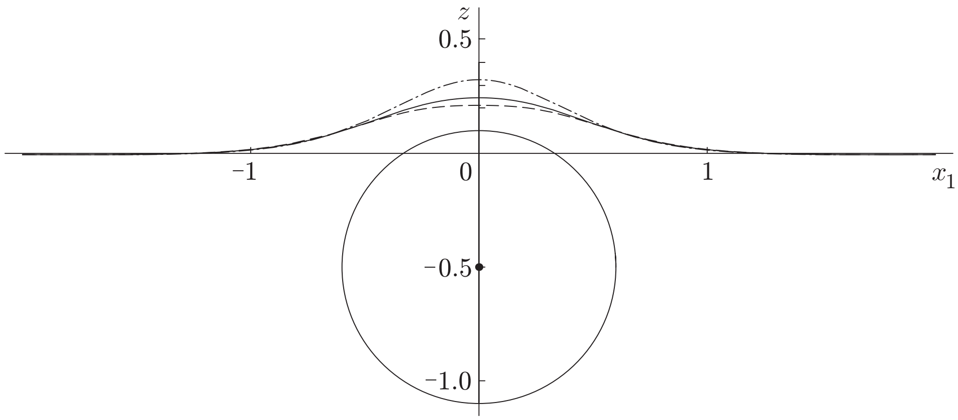

Fig. 2. Free surface in the case of vertical ascent of the sphere: the solid curve refers to the nonlinear modified dipole approximation, the dashed curve to the linear modified dipole approximation, and the dot-and-dashed curve to the dipole approximation.

The force acting on the sphere is determined by the formula

$$\boldsymbol{R} = -\int\limits_{S_\varepsilon} p\boldsymbol{n}\, dS,$$

where $\boldsymbol{n}$ is the external normal of the sphere. As the solution of system (2.1), (2.2), we seek the velocity potential, the pressure in the fluid, and the force $\boldsymbol{R}$ in the form of series in t

$$(\Phi, p, \boldsymbol{R}) = (\Phi_0, p_0, \boldsymbol{R}_0) + t(\Phi_1, p_1, \boldsymbol{R}_1) + (t^2/2)(\Phi_2, p_2, \boldsymbol{R}_2) + (t^3/3!)(\Phi_3, p_3, \boldsymbol{R}_3) + \ldots,$$

where

$$\boldsymbol{R}_i = -\int\limits_{S_\varepsilon} p_i\boldsymbol{n}\, dS.$$

Using the Cauchy–Lagrange integral, we obtain the following recurrent relations:

$$\begin{gathered} p_0(x,z) = -\Phi_1(x,z) - \mathrm{Fr}^{-2}z, \quad p_1(x,z) = -\Phi_2(x,z), \quad p_2(x,z) = -\Phi_3(x,z) - |\nabla\Phi_1(x,z)|^2, \\ p_3(x,z) = -\Phi_4(x,z) - 3\nabla\Phi_1(x,z)\cdot\nabla\Phi_2(x,z), \\ p_4(x,z) = -\Phi_3(x,z) - \nabla\Phi_1(x,z)\cdot\nabla\Phi_3(x,z) - |\nabla\Phi_2(x,z)|^2, \\ p_5(x,z) = -\Phi_6(x,z) - 10\nabla\Phi_2(x,z)\cdot\nabla\Phi_3(x,z) - 5\nabla\Phi_1(x,z)\cdot\nabla\Phi_4(x,z), \ \ldots\ . \end{gathered} \tag{6.1}$$

Using the expressions for h, φ, and ψ (see Sec. 4), we obtain $\Phi_0 = \Phi_2 = \Phi_4 = 0$. Then, formulas (6.1) yield $p_1 = p_3 = p_5 = 0$. Hence, we obtain $\boldsymbol{R}_1 = \boldsymbol{R}_3 = \boldsymbol{R}_5 = 0$. The coefficients $\boldsymbol{R}_0$, $\boldsymbol{R}_2$, and $\boldsymbol{R}_4$ of the force expansion are found by integration. For convenience of integration over the body surface, we pass to spherical coordinates with the origin at the center of the moving sphere:

$$x_1 = \varepsilon\cos\alpha\cos\beta + a_1t^2/2, \quad x_2 = \varepsilon\cos\alpha\sin\beta + a_2t^2/2, \quad z = \varepsilon\sin\alpha - 1 + a_3t^2/2,$$

$$-\pi/2 \leqslant \alpha \leqslant \pi/2, \qquad 0 \leqslant \beta \leqslant 2\pi.$$

We have

$$\boldsymbol{R}_0 = -\int\limits_{S_\varepsilon} p_0\boldsymbol{n}\, dS = -\int\limits_0^{2\pi}\int\limits_{-\pi/2}^{\pi/2} \tilde{p}_0(\varepsilon,\alpha,\beta,t)(\cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha)\varepsilon^2\cos\alpha\, d\alpha\, d\beta.$$

Note, the pressure p_0 on the sphere surface depends on time, since the sphere leaves its initial position while moving. Hence, the vector $\boldsymbol{R}_0$ is also time-dependent and can be expanded into a power series in t; this is a series in even powers, since the sphere-center coordinates depend quadratically on time:

$$\boldsymbol{R}_0 = \boldsymbol{R}_{00} + t^2\boldsymbol{R}_{02} + t^4\boldsymbol{R}_{04} + \ldots\ .$$

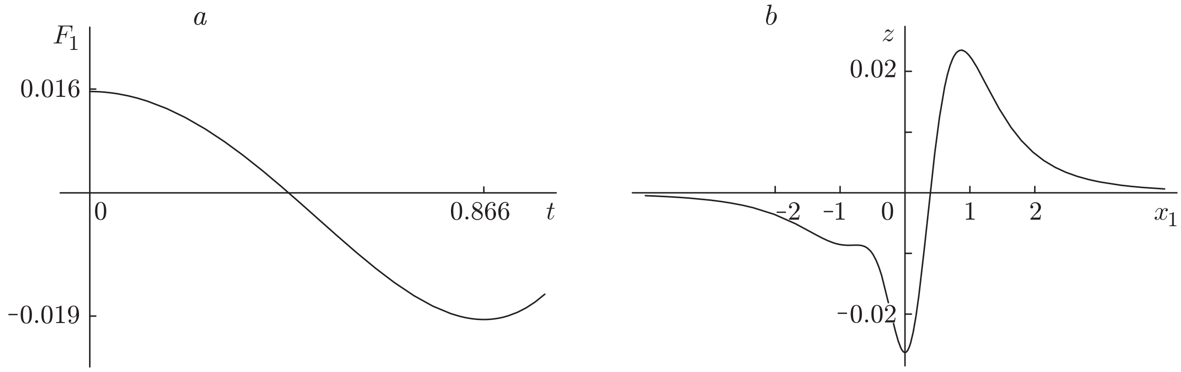

Fig. 3. Force directed oppositely to acceleration of the sphere versus time (a) and cross section ($x_2 = 0$) of the fluid surface at the time $t = 0.8661$ (b) for $\boldsymbol{A} = 0.25g$.

Analogously, we obtain

$$\boldsymbol{R}_2 = \boldsymbol{R}_{22} + t^2\boldsymbol{R}_{24} + \ldots, \qquad \boldsymbol{R}_4 = \boldsymbol{R}_{44} + t^2\boldsymbol{R}_{46} + \ldots .$$

The expansion of the force acting on the sphere in powers of t has the form

$$\boldsymbol{R} = \boldsymbol{R}_{00} + (\boldsymbol{R}_{02} + \boldsymbol{R}_{22}/2)t^2 + (\boldsymbol{R}_{04} + \boldsymbol{R}_{24}/2 + \boldsymbol{R}_{44}/24)t^4$$

$$+ (\boldsymbol{R}_6/6! + \boldsymbol{R}_{06} + \boldsymbol{R}_{26}/2 + \boldsymbol{R}_{46}/24)t^6 + (\boldsymbol{R}_7/7!)t^7 + \ldots .$$

Integration over the sphere yields

$$\boldsymbol{R} = \boldsymbol{F}_{\rm A} - 2\pi\varepsilon^3\boldsymbol{a}/3 + \boldsymbol{F},$$

where $\boldsymbol{F}_{\rm A}$ is the buoyancy force and $2\pi\varepsilon^3\boldsymbol{a}/3$ is the force acting on the sphere moving with an acceleration $\boldsymbol{a}$ in an infinite fluid. The term $\boldsymbol{F} = (F_1, F_2, F_3)$ is responsible for the wave load has the order ε^6:

$$F_1 = \pi a_1\varepsilon^6/12 - t^2\pi a_1\varepsilon^6({\rm Fr}^{-2} - 3a_3)/8 + t^4\pi a_1\varepsilon^6(2{\rm Fr}^{-4} - 15{\rm Fr}^{-2}a_3 + 75a_3^2)/96 + \ldots,$$

$$F_2 = \pi a_2\varepsilon^6/12 - t^2\pi a_2\varepsilon^6({\rm Fr}^{-2} - 3a_3)/8 + t^4\pi a_2\varepsilon^6(2{\rm Fr}^{-4} - 15{\rm Fr}^{-2}a_3 + 75a_3^2)/96 + \ldots,$$

$$F_3 = \pi a_3\varepsilon^6/6 + t^2\pi\varepsilon^6(-1 + 5a_3^2)/8 + t^4\pi\varepsilon^6(15{\rm Fr}^{-2} + 2(-6 + {\rm Fr}^{-4})a_3 - 33{\rm Fr}^{-2}a_3^2 + 48a_3^3)/48 + \ldots .$$

In the case of horizontal motion with slow acceleration of the sphere, the force directed oppositely to the motion depends on time nonmonotonically. The time when the force reaches an extremum (Fig. 3a) corresponds to the time when the second hump is formed on the free surface (see Fig. 1d). The cross section of the free surface by the vertical plane of symmetry is shown in Fig. 3b. In the case of fast horizontal motion, the wave load increases monotonically with time.

Thus, a small-time expansion of the free surface elevation is found in the case of uniformly accelerated motion of the sphere starting from rest in the dipole and modified dipole approximations. An expression for the force acting on the sphere is also obtained in the dipole approximation. The method used to reduce the initial problem to a system of integrodifferential boundary equations allows obtaining the solution of the problem with nonlinear conditions on the free surface.

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